603: Electromagnetic Theory I

Problem Sheet 1

(**1**) Making use of the result that $\nabla^2|\vec{r}-\vec{r}\,'|^{-1} = -4\pi\delta^3(\vec{r}-\vec{r}\,')$, show *by direct substitution* that if

$$\phi(\vec{r}\,) = \int \frac{\rho(\vec{r}\,')\, d^3\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\,, \tag{1}$$

then ϕ satisfies $\nabla^2\phi(\vec{r}\,) = -4\pi\,\rho(\vec{r}\,)$. (You need to be precise, and distinguish carefully between the integration variable $\vec{r}\,'$ and the variable $\vec{r}$.)

(**2a**) Writing $\vec{E}$ and $\vec{B}$ in terms of the gauge potentials ϕ and $\vec{A}$, obtain the equations for ϕ and $\vec{A}$ that result from plugging $\vec{E}$ and $\vec{B}$ into the two Maxwell field equations (with time dependence included). (Assume there is no medium present, so $\vec{D} = \vec{E}$ and $\vec{H} = \vec{B}$.)

(**2b**) It is possible to choose a gauge where $\vec{\nabla}\cdot\vec{A} + (1/c)\,\partial\phi/\partial t = 0$. (This is known as Lorenz gauge.) Show that in Lorenz gauge, the equations for ϕ and $\vec{A}$ that you obtained in part (2a) become simply wave operators acting on ϕ or $\vec{A}$, with ρ or $\vec{J}$ acting as source terms.

(**3a**) An infinite uniform line of charge, with (constant) charge λ per unit length, lies along the entire x axis from $-\infty$ to $+\infty$. A thin rigid rod of length $2L$ carries a uniform charge $-\lambda$ per unit length. Its centre is located at $z = a$ on the z axis, and the rod is constrained to lie in the (x, y) plane at $z = a$, making an angle α with the x axis. Write down, as a double integral over the points on the two lines of charge, the expression for the net force on the rod.

(**3b**) Evaluate first the integral over the points along the infinite line charge. Then evaluate the integral over the points along the rod, hence obtaining the expression for the net force on the rod. (A change of variable involving the tangent function can be useful for evaluating each of the integrals.)

(**3c**) What is the net force if the length L of the rod goes to infinity? What happens if α is zero?

Due in class on Tuesday 28th January

603: Electromagnetic Theory I

Problem Sheet 2

(**1**) By applying Green's theorem with $\phi(\vec{r}) = G(\vec{r}_1, \vec{r})$ and $\psi(\vec{r}) = G(\vec{r}_2, \vec{r})$, prove that a Dirichlet Green function is necessarily symmetric in its arguments; $G_D(\vec{r}_1, \vec{r}_2) = G_D(\vec{r}_2, \vec{r}_1)$.

(**2**) Suppose $G_N(\vec{r}, \vec{r}\,')$ is a Neumann Green function in a volume V bounded by the surface S, and that it is chosen to obey $\partial G_N(\vec{r}, \vec{r}\,')/\partial n' = -4\pi/A$ when $\vec{r}\,'$ lies on S, where $A = \int_S dS$ is the total area of the surface S. Observe that applying the same method as in (1a) no longer allows one to show that $G_N(\vec{r}, \vec{r}\,')$ must be symmetric in $\vec{r}$ and $\vec{r}\,'$. However, one can always modify the Neumann Green function to obtain one that *is* symmetric, by adding a properly-chosen harmonic function. Show this by carrying out the following steps:

Show that it is possible to add a carefully-chosen solution $F(\vec{r}, \vec{r}\,')$ of Laplace's equation $\nabla'^2 F(\vec{r}, \vec{r}\,') = 0$ to a non-symmetric Neumann Green function $G_N(\vec{r}, \vec{r}\,')$ to give a new Green function $\widetilde{G}_N(\vec{r}, \vec{r}\,') = G_N(\vec{r}, \vec{r}\,') + F(\vec{r}, \vec{r}\,')$ that *satisfies the same Poisson equation* $\nabla'^2\, \widetilde{G}_N(\vec{r}, \vec{r}\,') = -4\pi\, \delta^3(\vec{r} - \vec{r}\,')$ as $G_N(\vec{r}, \vec{r}\,')$ does, and *satisfies the same Neumann boundary condition*, and that *is* symmetric. **Note that you may choose a function $F(\vec{r}, \vec{r}\,')$ that does not depend at all on $\vec{r}\,'$.**)

(**3a**) Recall that for Robin boundary conditions $\frac{\partial \phi}{\partial n} + a\, \phi = f$ on the boundary S of a volume V, the function a was required to be everywhere non-negative on the boundary surface S. (See online lecture notes). The following example explicitly demonstrates that if the function a is not non-negative, then uniqueness need no longer hold. Take the volume V to be the interior of a sphere of radius R, centred on the origin. Obviously $\psi = cz = cr\cos\theta$ is a solution of $\nabla^2\psi = 0$, where c is any constant. Show that if a is chosen to be a specific negative constant (which you should determine explicitly) then $\psi = \phi_1 - \phi_2$ satisfies the Robin boundary condition $\partial\psi/\partial n + a\psi = 0$ on the boundary surface at $r = R$ for *any* value of c. This shows that the solution is not unique, and hence with this particular boundary condition the boundary-value problem is not well posed.

(**3b**) Consider instead $\psi = c\,(x^2 - y^2)$, and show that this satisfies Laplace's equation. Show that there exists a (different) specific negative constant a (calculate it) such that $\partial\psi/\partial n + a\psi = 0$ for any c.

Due in class on Tuesday 4th February

603: Electromagnetic Theory I

Problem Sheet 3

(**1**) Construct the Dirichlet Green function $G_D(x, y, z; x', y', z')$ for the infinite-volume region comprising the positive octant in Euclidean space, i.e. the region where $(x > 0, y > 0, z > 0)$. (This Green function can be used in order to solve the Dirichlet boundary-value problem in the positive octant, where the value of the potential is specified on the three orthogonal planes $x = 0$ and on $y = 0$ and on $z = 0$.)

Hint: First use the method of images to find the potential in the positive octant due to a charge q at $\vec{r}_1 = (x_1, y_1, z_1)$, in the presence of three orthogonal grounded planes at $x = 0$, at $y = 0$ and at $z = 0$. Think about all the mirror reflection points for three orthogonal mirrors.

(**2**) By making appropriate changes to the signs of the coefficients of the various terms you obtained in the expression for the Dirichlet Green function in question 1, construct the Neumann Green function $G_N(x, y, z; x', y', z')$ for the positive octant. (Be sure to demonstrate explicitly that your result for $G_N(x, y, z; x', y', z')$ does indeed satisfy the required boundary conditions.)

(**3a**) Eqn (2.38) in the current online lecture notes gives the general solution to the boundary-value problem for the solution of $\nabla^2\phi = 0$ in the upper half-space $z \ge 0$, subject to Dirichlet boundary conditions on the plane $z = 0$. Use this to write down the expression (leaving it as an integral), for the potential $\phi(x, y, z)$ in the upper half-space, subject to the boundary condition that $\phi(x', y', 0) = V$ (a constant) on the disk $x'^2 + y'^2 \le a^2$ and $\phi(x', y', 0) = 0$ when $x'^2 + y'^2 > a^2$ (outside the disk). (You may find it helpful to change to polar coordinates in the (x', y') plane.)

(**3b**) Evaluate the potential ϕ explicitly on the z axis, for $z > 0$.

(**3c**) Expand your result $\phi(0, 0, z)$ as a series in powers z, up to and including order z^5.

(**3d**) Expand $\phi(0, 0, z)$ instead as a series in $1/z$, up to and including order $1/z^4$.

Due on Tuesday 11th February

603: Electromagnetic Theory I

Problem Sheet 4

(1a) Consider the problem of the rectangular box of sides a, b and c, as discussed in section 3 of the on-line notes. (With potential 0 on five faces, and potential $V(x,y)$ on the face at $z = c$.) Calculate the coefficients A_{mn} in the expansion in equation (3.12) in the case that the potential $V(x,y)$ on the face at $z = c$ is is simply a constant, $V(x,y) = V_0$. (The integrals in (3.20) can easily be evaluated in this case.)

(1b) Find the expression (as a double infinite sum) for the potential evaluated at the centre of the box, i.e. find $\phi(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$. Express your answer as simply as possible.

(1c) Suppose now that the box is actually a cube, i.e. $a = b = c$. Construct a very simple argument, based on symmetry and the linear superposition of solutions, which gives the *exact* (and *very* simple!) expression for the potential at the centre of the cube. (Recall that the uniqueness theorem implies that if all the faces are at potential V_0, then all interior points will be at potential V_0.)

(1d) Evaluate your answer in part (1b) numerically, for the case $a = b = c$, keeping just the first few terms in the sum. Compare this with the exact result in part (1c).

(2a) Consider a two-dimensional potential theory problem in the (x,y) plane, where the solution to $(\partial^2/\partial x^2 + \partial^2/\partial y^2)\phi(x,y) = 0$ is required inside the semi-infinite U-shaped channel with sides $y \ge 0$ at $x = 0$; $y \ge 0$ at $x = a$; and $0 \le x \le a$ at $y = 0$. Suppose $\phi(x,y)$ obeys tha Dirichlet boundary condition that $\phi(0,y) = 0$ and $\phi(a,y) = 0$ for all $y \ge 0$; and that $\phi(x,0) = V(x)$ for $0 < x < a$, where $V(x)$ is a function to be specified later. Assume that $\phi(x,y)$ goes to zero as y goes to infinity.

Use the method of separation of variables to construct the solution for $\phi(x,y)$ inside the U-shaped channel, as an infinite series with coefficients expressed in terms of integrals involving $V(x)$.

(2b) Obtain the solution explicitly, in closed form, if $V(x) = V_0 \sin\frac{3\pi x}{a}$, where V_0 is a constant.

TURN OVER FOR PROBLEM 3!

(3) It will be useful later in the course to introduce 3-dimensional oblate spheroidal coordinates (μ, ν, φ), related to Cartesian coordinates by

$$x = a \cosh\mu \,\cos\nu\, \cos\varphi\,, \qquad y = a\cosh\mu\,\cos\nu\,\sin\varphi\,, \qquad z = a\sinh\mu\,\sin\nu\,,$$

where a is a constant. It can be shown that the Laplacian in oblate spheroidal coordinates is given by

$$\begin{aligned}\nabla^2\Psi &= \frac{1}{a^2(\cosh^2\mu - \cos^2\nu)}\Big[\frac{1}{\cosh\mu}\frac{\partial}{\partial\mu}\Big(\cosh\mu\,\frac{\partial\Psi}{\partial\mu}\Big) + \frac{1}{\cos\nu}\frac{\partial}{\partial\nu}\Big(\cos\nu\,\frac{\partial\Psi}{\partial\nu}\Big)\Big] \\ &\quad + \frac{1}{a^2\cosh^2\mu\,\cos^2\nu}\,\frac{\partial^2\Psi}{\partial\varphi^2}\,. \qquad (1)\end{aligned}$$

Use this to show that the Laplace equation $\nabla^2\Psi = 0$ is separable in oblate spheroidal coordinates, by looking for factorised solutions of the form $\Psi(\mu,\nu,\varphi) = f(\mu)\,h(\nu)\,P(\varphi)$. Give the ordinary differential equations for f, h and P explicitly. (You will need to introduce two independent separation constants.) **Note:** You are not asked to *solve* the equations, but just to derive the differential equations for f, h and P.

(4) Optional, for self-edification only:

Show that the Laplacian in oblate spheroidal coordinates is indeed given by equation (1). This can be done either by pedestrian means, using the chain rule for differentiation and grinding through the calculation, or much more easily and elegantly if you are familiar with general-coordinate-tensor analysis.

If you do want to do the calculation by the pedestrian approach, it is probably easiest to work backwards, taking (1) as the starting point and then converting from $\partial/\partial\mu$, $\partial/\partial\nu$ and $\partial/\partial\varphi$ derivatives to $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ derivatives using the chain rule. The goal then is to show that (1) becomes simply $\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}$. The calculation is straightforward, but rather tedious. Systematic organisation of the calculation, and using abbreviations for frequently-occurring quantities, can be very helpful here!

Due on Tuesday 18th February

603: Electromagnetic Theory I

Problem Sheet 5

(1) Consider again the problem of calculating the potential everywhere outside a spherical surface of radius a, subject to the boundary condition that the upper hemisphere of the $r = a$ surface is held at potential $+V$, and the lower hemisphere at potential $-V$, where V is a constant. Use the following procedure for obtaining the solution in the form $\phi(r, \theta) = \sum_{\ell \geq 0} B_\ell \, r^{-\ell-1} \, P_\ell(\cos\theta)$:

(1a) Use eqn (4.68) in the lectures to obtain an expression for B_ℓ as an integral involving $P_\ell(x)$ integrated between $x = 0$ and $x = 1$. (You can use $P_\ell(x) = (-1)^\ell \, P_\ell(-x)$ to replace an integral over negative x by an integral over positive x.)

(1b) Let $W(t) = \int_0^1 dx (1 - 2xt + t^2)^{-1/2}$ be the integral of the formula (4.40) for the generating function. Show how the evaluation of the integrals in Qu. (1a) can be reduced to the exercise of obtaining the coefficients in the power-series expansion of $W(t)$ in powers of t.

(1c) Expand the series for $W(t)$ explicitly, up to and including order t^5, and show that this gives the same result as the first three terms in the expression (4.84) for the potential, which was obtained using off-axis extrapolation.

(2a) Make the necessary changes in the calculation of the Dirichlet Green function in section 2.5.2 in order to obtain the analogue of eqn (2.59) for the potential *inside* a spherical surface at $r = a$ on which Dirichlet boundary conditions are specified.

(2b) Apply this result to the case where the upper hemisphere at $r = a$ is held at constant potential $+V$ and the lower hemisphere at $-V$. Obtain the explicit closed-form expression for $\phi(z)$ on the z axis for $|z| < a$.

(2c) Use the method of off-axis extrapolation to obtain the first three terms in the expansion of the potential $\phi(r, \theta)$, and show that it takes the form given in eqn (4.90).

(2d) Show that these terms are indeed related by inversion to the corresponding terms in the $r > a$ expansion in eqn (4.84).

Due on Thursday 27th February

603: Electromagnetic Theory I

Problem Sheet 6

(1a) A thin, flat disc of radius a is located in the (x, y) plane at $z = 0$, with its centre at the origin. It has a surface charge density $\sigma = k(a^2 - \rho^2)^{-1/2}$, where ρ is the distance out from the centre of the disc and k is a constant. Calculate (by elementary methods) the potential on the z axis, both for positive z and negative z, showing that it can be written as

$$\phi(z) = 2\pi k \arctan \frac{a}{|z|}\,.$$

(1b) Show, by using the method of off-axis extrapolation, that the potential at points (r, θ, φ) with $r > a$ is given by

$$\phi(r, \theta) = \frac{2\pi k a}{r} \sum_{n \geq 0} \frac{(-1)^n}{2n + 1} \left(\frac{a}{r}\right)^{2n} P_{2n}(\cos\theta)$$

(1c) Show that the on-axis potential can be re-written as $\phi(z) = \pi^2 k - 2\pi k \arctan(|z|/a)$. Using this, find the analogous off-axis extrapolation to give $\phi(r, \theta)$ that is valid for $r < a$. Do this both for the region above the plane of the disc and the region below the plane of the disc.

(1d) Compare your expansions for $\phi(r, \theta)$ in Qu. (1b) and (1c), and discuss whether or not they are related by inversion.

(2a) Use your result in Qu. (1c) to calculate the potential at all points on the surface of the disc. (Recall that $P_\ell(x) = (-1)^\ell P_\ell(-x)$, so $P_\ell(0) = 0$ if ℓ is odd.)

(2b) Calculate the total charge on the disc. Hence, making use of your result from part (2a), calculate the capacitance of a thin conducting disc of radius a.

TURN OVER FOR QUESTION 3!

(**3a**) Recall that the oblate spheroidal coordinates (μ, ν, φ) that were introduced in homework 4 are related to Cartesian coordinates by

$$x = a \cosh\mu \, \cos\nu \, \cos\varphi \,, \qquad y = a \cosh\mu \, \cos\nu \, \sin\varphi \,, \qquad z = a \sinh\mu \, \sin\nu \,.$$

Show that the surface $\mu =$ constant is an ellipsoid of revolution around the z axis,

$$\frac{x^2 + y^2}{\cosh^2\mu} + \frac{z^2}{\sinh^2\mu} = a^2 \,.$$

What are the lengths of the semi-major and semi-minor axes? Show that the surface $\mu = 0$ represents a degeneration of the ellipsoid to a thin disc of radius a, lying in the $z = 0$ plane and centered on the origin.

(**3b**) Hence show, by making use of the Laplacian in oblate spheroidal coordinates of homework 4, and considering a potential $\Phi(\mu)$ that **depends only on μ, but not ν or** φ, that the potential everywhere outside a thin charged conducting disc of radius a lying in the $z = 0$ plane and centred on the origin, and held at potential V, is given in oblate spheroidal coordinates by

$$\Phi = V - \frac{2V}{\pi} \arctan \sinh\mu \,.$$

(**3c**) By expressing μ in terms of the Cartesian coordinates (x, y, z), and hence in terms of spherical polar coordinates, show that the potential outside the disc can be written in closed form as

$$\Phi = V - \frac{2V}{\pi} \arctan\left[\frac{1}{\sqrt{2}\,a}\left(r^2 - a^2 + \sqrt{(r^2 - a^2)^2 + 4a^2 r^2 \cos^2\theta}\right)^{1/2}\right].$$

If you have access to algebraic computing, you can easily verify for the first few terms in a small-r expansion of this expression that they agree with the terms in the expansion in Qu. (1c) above.

Due on Tuesday 17th March.

603: Electromagnetic Theory I

Commentary on Problem Sheet 6

This problem sheet was concerned entirely with studying the problem of finding the potential everywhere outside a thin conducting disk of radius a that is held at a (constant) potential V.

(1): Solution by off-axis extrapolation

The approach taken in Question 1 has as its starting point a problem that is ostensibly unrelated to the conducting disk, namely solving for the potential due to a particular surface charge density on the disk. The charge density specified in the problem is not, of course, just some random function; it has been chosen precisely because it is in fact the charge density that arises on a conducting disk that is held at a fixed (constant) potential.

It is very easy to solve for the potential on the z axis, given this charge distribution on the disk, by elementary means, and this is done in Qu. (1a). The answer that you are asked to find in Qu. (1a) lends itself to being expanded as a power series in inverse powers of z, and then, in Qu. (1b) you are asked perform the off-axis extrapolation to obtain the expression for the potential $\phi(r,\theta)$ for $r > a$ as an expansion in terms of Legendre polynomials. This gives the result

$$\phi(r,\theta) = \frac{2\pi k a}{r} \sum_{n\ge 0} \frac{(-1)^n}{2n+1} \Big(\frac{a}{r}\Big)^{2n} P_{2n}(\cos\theta)\,, \qquad r > a\,. \tag{1}$$

Note that this is valid both in the upper half-space where $0 \le \theta < \frac{1}{2}\pi$, and in the lower half-space where $\frac{1}{2}\pi < \theta \le \pi$. This is evident from the fact that (1) has the property $\phi(r,\theta) = \phi(r,\pi-\theta)$. (The potential must obviously be the same at any point in the upper half-space and its mirror-reflection point in the lower half-space.)

In Qu. (1c) you are asked to use off-axis extrapolation to find the series expansion for $\phi(r,\theta)$ in the region $r < a$. To do this, it is useful first to re-express the potential on the z axis in a form that lends itself to performing a Taylor expansion in (non-negative) powers of z. The result you should then find for the off-axis extrapolation is

$$\phi(r,\theta) = \pi^2\, k - 2\pi k \sum_{n\ge 0} \Big(\frac{r}{a}\Big)^{2n+1} P_{2n+1}(|\cos\theta|)\,, \qquad r < a\,. \tag{2}$$

Note that we have placed an absolute value sign around the $\cos\theta$ argument of the Legendre polynomials. This is what you find if you are careful about the off-axis extrapolation in the lower half-space. It ensures that the potential in the $r < a$ region is again reflection-symmetric across the $z = 0$ plane, as it must be.

In Qu. (1d) you are asked to comment on whether the potentials in eqn (1) and eqn (2) obey the inversion symmetry discussed in the lectures. The answer is clearly no, since the expansion in eqn (1) involves only the $P_\ell(\cos\theta)$ with ℓ being even, whereas the expansion in eqn (2) involves (almost entirely) the $P_\ell(\cos\theta)$ with ℓ being odd. However, see later, for further a discussion!

(2): Recognising that the potential is constant on the disk

Until now, there has been no reason to suspect that one is secretly solving the problem of the potential outside a thin conducting disk. However, in Qu. (2a) you are asked to use eqn (2) to find the potential at all points on the surface of the disk. In other words, calculate $\phi(r, \frac{1}{2}\pi)$ using eqn (2). Obviously, one must use eqn (2) rather than eqn (1) to do this, since points on the disk all have $r \le a$. Noting that at $\theta = \frac{1}{2}\pi$ one has $\cos\theta = 0$, and that $P_\ell(0) = 0$ when ℓ is odd, we see that the answer is that on the surface of the disk

$$\phi(r, \tfrac{1}{2}\pi) = \pi^2\, k\,. \tag{3}$$

That is to say, seemingly miraculously, the surface of the disk is at a constant potential $V = \pi^2 k$. In other words, all along we have been solving the problem of the potential outside a thin conducting disk. The constant k in the original problem can now be written as $k = V/\pi^2$.

It is now an easy matter to calculate in Qu. (2b) the capacitance of the disk, by integrating the original charge density on the disk to find its total charge Q, and then calculating $C = Q/V$.

(3): Closed form expression for the potential

In Qu. (3), one discovers that the potential outside a thin conducting disk can in fact be solved explicitly in closed form. (That is to say, as an explicit elementary function of r and θ, rather than as the infinite series we have seen until now.) This is done by making use of the oblate spheroidal coordinate system that was introduced in homework 4. These coordinates (μ, ν, φ) are related to the Cartesian coordinates (x, y, z) by

$$x = a\, \cosh\mu\, \cos\nu\, \cos\varphi\,, \qquad y = a\, \cosh\mu\, \cos\nu\, \sin\varphi\,, \qquad z = a\, \sinh\mu\, \sin\nu\,. \tag{4}$$

In Qu. (3a) you are asked to show that these obey the relation

$$\frac{x^2+y^2}{\cosh^2\mu} + \frac{z^2}{\sinh^2\mu} = a^2\,. \tag{5}$$

Thus a surface of constant μ has the form of an ellipsoid of revolution:

$$\frac{x^2+y^2}{\alpha^2} + \frac{z^2}{\beta^2} = 1\,, \tag{6}$$

where

$$\alpha = a\, \cosh\mu\,, \qquad \beta = a\, \sinh\mu\,, \tag{7}$$

and so the length of the semi-major axis is $\alpha = a\, \cosh\mu$ and the length of the semi-minor axis is $\beta = a\, \sinh\mu$. (Clearly $\alpha \ge \beta$.)

When μ gets large, the constant μ surfaces become larger and larger, and also more and more nearly spherical, since α and β become more and more nearly the same (*i.e.*

$\beta/\alpha = \tanh\mu \to 1$). As μ goes to zero the ratio $\beta/\alpha = \tanh\mu$ goes to zero, so the ellipsoid of revolution becomes more and more flattened, ultimately becoming just a thin disk of radius a squashed flat on the $z = 0$ plane.

In Qu. (3b), you are asked to solve for a potential $\Phi(\mu)$ that is a function only of the μ coordinate in oblate spheroidal coordinates. This is the analogue of solving for a potential $\Psi(r)$ that is a function only of r in spherical polar coordinates. In that case, the potential $\Psi(r)$ is obviously constant on surfaces of constant r (*i.e.* on spheres). In the present case, the potential $\Phi(\mu)$ will be constant on surfaces of constant μ (*i.e.* on the ellipsoids of revolution that were found previously). In particular, the $\mu = 0$ surface is the thin disk of radius a. Thus, if we solve for the general potential $\Phi(\mu)$ that satisfies $\nabla^2\Phi = 0$, then by matching the constants of integration appropriately, we will be able to obtain the expression for the potential everywhere outside a thin disk of radius a, held at (constant) potential V.

The Laplacian in oblate spheroidal coordinates was found in homework 4. If the potential depends only on μ, then this implies that

$$\nabla^2\Phi(\mu) = \frac{1}{a^2\,(\cosh^2\mu - \cos^2\nu)\,\cosh\mu}\,\frac{d}{d\mu}\Big(\cosh\mu\,\frac{d\Phi(\mu)}{d\mu}\Big)\,, \tag{8}$$

and so $\nabla^2\Phi = 0$ implies

$$\frac{d}{d\mu}\Big(\cosh\mu\,\frac{d\Phi(\mu)}{d\mu}\Big) = 0\,. \tag{9}$$

It is easy to see that the general solution can be written as

$$\Phi = b + c\,\arctan\sinh\mu\,, \tag{10}$$

where b and c are constants. Now, we want the potential on the $\mu = 0$ surface (the disk) to be equal to V, and we want the potential to go to zero at large distances (*i.e.* as μ goes to infinity). These two conditions determine b and c, leading to the final answer

$$\Phi = V - \frac{2V}{\pi}\,\arctan\sinh\mu\,. \tag{11}$$

Finally, to re-express the potential Φ in terms of the spherical polar coordinates, we just have to solve eqn (5) for μ in terms of x, y and z and then express x, y and z in terms of the spherical polar coordinates. This produces the expression that you are asked to obtain in Qu. (3c).

Further remarks about inversion

If you look back at the lecture notes on the topic of inversion symmetries, you will see that the essential idea is that if one knows that the potentials in the regions $r \ge a$ and $r \le a$ are of the forms

$$\phi_>(r,\theta) = \sum_{\ell\ge0} B_\ell\, r^{-\ell-1}\, P_\ell(\cos\theta)\,, \qquad \phi_<(r,\theta) = \sum_{\ell\ge0} A_\ell\, r^\ell\, P_\ell(\cos\theta) \tag{12}$$

respectively, then by matching them up at $r = a$ and using the fact that the $P_\ell(\cos\theta)$ form a complete set in terms of which any smooth function $f(\theta)$ on the sphere can be expanded, then one must have

$$B_\ell = a^{2\ell+1}\, A_\ell\,. \tag{13}$$

Now, on the face of it we might seem to be satisfying these conditions in the present problem, of the conducting disk. However, although the expression (1) in the $r > a$ region is certainly of the form given for $\phi_>(r,\theta)$ in (12), the expression (2) in the $r < a$ region is not exactly of the form $\phi_<(r,\theta)$. This is because of the absolute value sign around the $\cos\theta$ argument of the Legendre polynomial $P_{2n+1}(|\cos\theta|)$, which means that $\phi_<(r,\theta)$ is not a smooth function of θ on the sphere. (Just like the function $|x|$, which, although continuous, has a discontinuity in its gradient at $x = 0$.) One cannot therefore make a naive matching of the potentials in eqn (1) and eqn (2) at $r = a$, and expect the coefficients in the $r > a$ expansion (1) to match the with coefficients in the $r < a$ expansion (2) as in (13). And sure enough, they don't.

Suppose, though, we try something a bit less naive. If we set $r = a$ in eqn (1) we get

$$\phi(a,\theta) = \frac{2V}{\pi} \sum_{n\ge0} \frac{(-1)^n}{2n+1}\, P_{2n}(\cos\theta)\,, \qquad r \to a^+ \tag{14}$$

(after using $k = V/\pi^2$, which we learned earlier).[1] This series can be summed in closed form, to give[2]

$$\phi(a,\theta) = \frac{2V}{\pi}\ \arctan\sqrt{\frac{1}{|\cos\theta|}}\,, \qquad r \to a^+\,. \tag{15}$$

On the other hand, using the expression (2) can be used to write $\phi(r,a)$ as $r \to a^-$ (*i.e.* approaching a from below), and this too can be summed, giving

$$\phi(a,\theta) = V - \frac{2V}{\pi}\ \arctan\sqrt{|\cos\theta|}\,, \qquad r \to a^-\,. \tag{16}$$

Finally, we can observe, using the basic properties of the arctangent function, that the expressions (15) and (16) are identical. Thus, invoking the uniqueness theorem, which assures us that there is one and only one solution to Laplace's equation subject to well-posed boundary conditions, we see that the solution (1) for the potential outside $r = a$ can be used to deduce the boundary values at $r = a$ and that these uniquely lead to the extrapolation given by eqn (2) inside $r = a$. Thus there is, in this sense, an inversion relation between the $r > a$ and the $r < a$ solutions. It's just that it is not the naive one of the form (13) that one might originally have expected. The entire reason for the non-triviality of

[1]The notation $r \to a^+$ denotes that r is being sent to a *from above*, and hence the expression for the potential in the region $r > a$ is being used.

[2]We leave the summation of the series as an exercise for the reader. For example, consider $\int_0^1 dt[G(x,it)+G(-x,it)]$, where $G(x,t) = (1-2xt+t^2)^{-1/2}$ is the generating function for the Legendre polynomials.

the inversion relation in this case is that the function describing the potential in the region $r < a$ is not a smooth function of θ.

It is worth remarking that the expressions (15) and (16) for the potential at $r = a$ can also be straightforwardly found from the closed-form expression for Φ that you were asked to obtain in Qu. (3c) of homework 6.

Finally, a remark about why the potential in the region $r > a$ given by eqn (1) is a smooth function of θ whereas the potential in the region $r < a$ given by eqn (2) is not. In the region $r > a$ there is no obstruction to passing through the $z = 0$ plane at $\theta = \frac{1}{2}\pi$, and so the potential for $0 \le \theta < \frac{1}{2}\pi$ must join smoothly onto the potential for $\frac{1}{2}\pi < \theta \le \pi$. On the other hand, in the region $r < a$ there *is* an obstruction to passing smoothly through the $z = 0$ plane; there is a surface charge density localised in the $z = 0$ plane for $r < a$. This reflects itself in the fact that the electric field normal to the disk points in opposite directions just above the disk and just below the disk. (And in turn, the gradient of the scalar potential has the opposite sign just above and just below the disk, as is implied by the absolute value sign in the expression (2) for the potential in the region $r < a$.)

Notice that even before doing any detailed calculations, the fact that the potential in the region $r > a$ must clearly be a smooth function of the r and θ coordinates, combined with the fact that this potential must obviously obey the reflection symmetry $\phi(r,\theta) = \phi(r,\pi-\theta)$, immediately tells us that its expansion must be of the form

$$\phi(r,\theta) = \sum_{p\ge 0} B_{2p}\, r^{-2p-1}\, P_{2p}(\cos\theta)\,, \tag{17}$$

for some as-yet undetermined coefficients B_{2p}. (That is, the coefficients B_{2p+1} in a general expansion $\phi(r,\theta) = \sum_{\ell\ge 0} B_\ell\, r^{-\ell-1}\, P_\ell(\cos\theta)$ must necessarily be zero.)

In the region $r < a$, on the other hand, we can make no analogous *a priori* prediction about the general form of the expansion, because the coefficients A_ℓ in a general expansion

$$\phi(r,\theta) = \sum_{\ell\ge 0} A_\ell\, r^\ell\, P_\ell(\cos\theta) \tag{18}$$

could be different in the upper half-space and the lower half-space. In fact, as we know from the detailed calculations, the coefficients A_ℓ^+ in the upper half-space and A_ℓ^- in the lower half-space obey

$$\begin{aligned} A_0^+ &= A_0^- = V\,, \\ A_{2p}^+ &= A_{2p}^- = 0\,, \qquad p = 1,2,\ldots\,, \\ A_{2p+1}^+ &= -A_{2p+1}^-\,, \qquad p = 0,1,2,\ldots\,. \end{aligned} \tag{19}$$

But without doing the detailed calculation, while we could certainly predict on the grounds of the reflection symmetry that $A_{2p}^+ = A_{2p}^-$ and $A_{2p+1}^+ = -A_{2p+1}^-$, there doesn't appear to be any general argument that would predict that (most of) the A_{2p} coefficients would be zero.

603: Electromagnetic Theory I

Problem Sheet 7

(**1**) The potential on the surface $r = a$ is given to be

$$\phi(a, \theta, \varphi) = V_0 \sin\theta \cos\theta \sin\varphi\,. \tag{1}$$

Obtain two fully explicit expressions for $\phi(r, \theta, \varphi)$, valid in the two regions $r > a$ and $r < a$ respectively. (Hint: Using the expressions for the first few spherical harmonics given in the lectures, first express the boundary value (1) in terms of the $Y_{\ell m}(\theta, \varphi)$.)

(**2a**) In Cartesian coordinates, show that if $\vec{p}$ is a constant vector, then $\phi \equiv \vec{p} \cdot \vec{\nabla}(1/r)$ satisfies Laplace's equation for $r > 0$.

(**2b**) Obtain the expression for the potential ϕ in part (2a) as a fully explicit expansion in spherical harmonics. [**Note:** The answer involves just a *finite number* (small!) of terms. The coefficients can be given completely explicitly.]

(**3**) This problem is concerned with establishing some properties of the spherical harmonics:

(**3a**) Consider the angular momentum generators $L_1 = -\mathrm{i}(y\partial_z - z\partial_y)$, $L_2 = -\mathrm{i}(z\partial_x - x\partial_z)$, $L_3 = -\mathrm{i}(x\partial_y - y\partial_x)$, (where $\partial_x \equiv \partial/\partial_x$, etc.) Using the relations $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, show that

$$L_\pm = \pm e^{\pm\mathrm{i}\varphi}\Big(\frac{\partial}{\partial\theta} \pm \mathrm{i}\cot\theta\frac{\partial}{\partial\varphi}\Big)\,, \qquad L_3 = -\mathrm{i}\frac{\partial}{\partial\varphi}\,,$$

where $L_\pm \equiv L_1 \pm \mathrm{i}\, L_2$.

(**3b**) Using the generalised Rodrigues formula for the associated Legendre functions P_ℓ^m, and the definition of the spherical harmonics, derive expressions for $L_3 Y_{\ell m}$, $L_+ Y_{\ell m}$ and $L_- Y_{\ell m}$. (In each case, the answer is a constant factor times a *single* spherical harmonic.)

Due on Tuesday 31 March (submit single PDF to grader by e-mail)

603: Electromagnetic Theory I

Problem Sheet 8

(**1**) This problem is concerned with establishing some properties of the Bessel functions $J_n(x)$ which are needed when solving Laplace's equation in cylindrical polar coordinates. These can be derived from a generating function, which defines the $J_n(x)$ (for n an integer) as follows:

$$G(x,t) \equiv e^{\frac{1}{2}x(t-t^{-1})} = \sum_{n=-\infty}^{\infty} t^n J_n(x) \tag{1}$$

(**1a**) Use eqn (1) to show that $J_n(x)$ as defined above does indeed satisfy Bessel's equation $x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0$. (The proof is analogous to the one given in the lectures for the generating function for the Legendre polynomials.)

(**1b**) Use eqn (1) to show the three identities

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x),\;\; J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x),\;\; J_{-n}(x) = (-1)^n J_n(x).$$

(**2**) A hollow semi-infinite conducting cylinder of radius a extends from $z = 0$ to $z = \infty$, with its axis coinciding with the z axis. The cylinder is grounded, and the potential on the disk $0 \le \rho \le a$ forming its base at $z = 0$ obeys the boundary condition

$$\phi(\rho,\varphi,z)\Big|_{z=0} = V\, J_1(\frac{b\rho}{a})\, \sin\varphi\,,$$

where V is a constant and b is the first positive zero of the Bessel function $J_1(x)$ (i.e. the smallest positive constant for which $J_1(b) = 0$). Calculate the potential $\phi(\rho,\varphi,z)$ everywhere inside the cylinder (i.e. $0 \le \rho \le a$ and $z \ge 0$). (The boundary condition at $z = 0$ has been chosen so that the answer is a simple closed-form expression.)

(**3**) Consider the Dirichlet Green function in the unbounded space between the planes $z = 0$ and $z = h$. Using the same approach as was discussed in the lectures in sections 4.9 and 5.5, show that the required Green function can be written as

$$G(\vec r, \vec r\,') = 2 \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\varphi-\varphi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh(k(h-z_>))}{\sinh(kh)}\,, \tag{2}$$

where $z_<$ is the lesser, and $z_>$ is the greater, of z and z'. Thus you need to prove that (2) obeys the proper Dirichlet boundary conditions for this problem, and that it obeys $\nabla^2 G(\vec r, \vec r\,') = -4\pi\delta^3(\vec r - \vec r\,')$.

Due on Tuesday 7th April (submit single PDF to grader by e-mail)

Turn over for some formulae and hints...

You may use without proof the completeness relation

$$\int_0^\infty J_m(k\rho) J_m(k\rho')kdk = \frac{1}{\rho}\,\delta(\rho - \rho')\,. \tag{3}$$

The delta function $\delta(\varphi - \varphi')$ will come from a standard completeness relation in Fourier analysis. The delta function $\delta(z - z')$ will come from the discontinuity in the gradient of the right-hand side of eqn (2) as z passes from $z < z'$ to $z > z'$

603: Electromagnetic Theory I

Problem Sheet 9

(**1**) Calculate the monopole Q, dipole p_i and quadrupole Q_{ij} moments, defined in eqn (6.30) of the lectures, for each of the following two charge distributions:

(**1a**) Charge q at $\vec{r} = (a, 0, 0)$; charge $-q$ at $\vec{r} = (-a, 0, 0)$; charge q at $\vec{r} = (0, a, 0)$; and charge $-q$ at $\vec{r} = (0, -a, 0)$. (i.e. four point charges in total.)

(**1b**) Charge q at $\vec{r} = (0, 0, a)$; charge q at $\vec{r} = (0, 0, -a)$; and charge $-2q$ at $\vec{r} = (0, 0, 0)$. (i.e. three point charges in total.)

(**2**) A charge distribution is given in spherical polar coordinates by

$$\rho = \frac{1}{64\pi}\, r^2 e^{-r}\, \sin^2\theta\, .$$

Calculate all its multipole moments $q_{\ell m}$ (as defined by eqn (6.46) in the lectures).

(Only a few of the $q_{\ell m}$ are non-zero. Obtain explicit expressions for them all. As usual in such cases, it is probably easiest to do the calculation by expressing the given function (i.e. the charge density ρ) explicitly in terms of spherical harmonics before plugging it into the integral, since then one can trivially use the known orthogonality integrals for the spherical harmonics to read off the results.)

(**3a**) A localised charge distribution has monopole, dipole and quadrupole moments Q, p_i and Q_{ij} for some given choice of origin for Cartesian coordinates x_i. With respect to another system x'_i of Cartesian coordinates, parallel to the x_i but with origin located at $x_i = a_i$ (a_i are given constants), the corresponding multipole moments are Q', p'_i and Q'_{ij}. Obtain explicit expressions for Q', p'_i and Q'_{ij} in terms of Q, p_i, Q_{ij} and the displacement vector a_i.

(**3b**) Hence deduce that, up to the order in multipole moments that you have calculated here, the *first non-vanishing multipole moment* is independent of the choice of origin, but the higher multipole moments depend upon the choice of origin.

(In other words, if the net charge Q is non-zero then it is independent of the choice of origin but the dipole and quadrupole moments do depend on the choice of origin. If the charge Q vanishes, then the dipole moment is independent of the choice of origin but the quadrupole moment does depend on the choice of origin. If the charge Q and the dipole moment both vanish, then the quadrupole moment does not depend on the choice of origin.)

Continues with Question 4 on next page...

(**4**) The pattern seen in question 3 continues to hold for all the higher multipole moments too. That is to say, the first non-vanishing multipole moment is independent of the choice of origin, but the higher multipole moments do then depend on the choice of origin. Show explicitly that if a localised charge distribution $\rho(\vec{r})$ has vanishing monopole, dipole and quadrupole moments, then the values of the 7 independent components Q_{ijk} of the octopole moment are independent of the choice of origin.

Note that this octopole result is the first "non-trivial" case to prove, in the sense that one now has to be rather careful about the trace terms. In particular, the fact that the quadrupole moment vanishes means that one knows $\int (3x_i\, x_j - r^2\, \delta_{ij})\, \rho(\vec{r})\, d^3x = 0$; it does NOT mean that one can say that $\int x_i\, x_j\, \rho(\vec{r})\, d^3x = 0$.

Due on Tuesday 14th April (submit single PDF to grader by e-mail)

603: Electromagnetic Theory I

Commentary on Problem Sheet 9

(1) Calculation of multipole moments for some simple arrays of point charges:

These two examples are both quite straightforward. For example, in (1a) one just needs to write down the charge density for the given array of charges, namely

$$\rho(\vec{r}\,) = q\,[\delta(x-a) - \delta(x+a)]\,\delta(y)\,\delta(z) + q\,[\delta(y-a) - \delta(y+a)]\,\delta(x)\,\delta(z)\,, \tag{1}$$

and then plug in to the formulae for Q, p_i and Q_{ij}. Because of the explicit nature of the charge distribution, there is really nothing for it but to work out all the components one by one: Q, p_1, p_2, p_3, Q_{11}, Q_{12}, etc.

2) Calculation of multipole moments for the given charge distribution:

A straightforwad matter of plugging into the formula giving the multipole moments in the $q_{\ell m}$ notation. As is usually the case when working with an explicit example where one can recognise how to write the function easily using the spherical harmonics, here it is worth first of all expressing the given $\rho(r,\theta,\varphi)$ in terms of the spherical harmonics, because then it is very easy to evaluate the integrals giving $q_{\ell m}$. In this example we may note that the $\sin^2\theta$ angular dependence may be expressed in terms of just

$$Y_{0,0}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \qquad \text{and} \qquad Y_{2,0}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)\,. \tag{2}$$

3) Calculate monopole, dipole and quadrupole moments with respect to a displaced origin:

Letting $x_i' = x_i + a_i$, we calculate

$$\begin{aligned} Q' &= \int \rho(\vec{r}\,)\, d^3\vec{r}\,, \\ p_i' &= \int x_i'\, \rho(\vec{r}\,)\, d^3\vec{r} = \int (x_i + a_i)\, \rho(\vec{r}\,)\, d^3\vec{r} = \int x_i\, \rho(\vec{r}\,)\, d^3\vec{r} + a_i \int \rho(\vec{r}\,)\, d^3\vec{r}\,, \qquad (3) \\ Q_{ij}' &= \int [3x_i'\, x_j' - x_k'\, x_k'\, \delta_{ij}]\, \rho(\vec{r}\,)\, d^3\vec{r} \\ &= \int [3(x_i + a_i)(x_j + a_j) - (x_k + a_k)(x_k + a_k)\, \delta_{ij}]\, \rho(\vec{r}\,)\, d^3\vec{r} = \cdots\,. \qquad (4) \end{aligned}$$

Hence we obtain expressions for Q', p_i' and Q_{ij}' in terms of Q, p_i, Q_{ij} and a_i. (E.g. $Q' = Q$, $p_i' = p_i + Q\, a_i$, etc.)

In part (3b), one now easily verifies that the statements are true, up to the levels calculated in (3a), namely that for a given charge density the first non-zero multipole moment is independent of the choice of origin (i.e. does not depend on the vector a_i), while the higher multipole moments do depend on a_i.

4) More of the same, but now non-trivial for the octopole:

The octopole provides the first non-trivial example in this sequence. So one has to to calculate Q'_{ijk}, given by

$$Q'_{ijk} = \int \Big[5(x_i+a_i)(x_j+a_j)(x_k+a_k) - [(x_i+a_i)\,\delta_{jk} + (x_j+a_k)\delta_{ik} + (x_k+a_k)\delta_{ij}]\,(x_\ell+a_\ell)(x_\ell+a_\ell)\Big]\,\rho(\vec{r}\,)\,d^3\vec{r}\,, \tag{5}$$

and show how it can be expressed in terms of Q, p_i, Q_{ij}, Q_{ijk} and a_i. It is good to organise the calculation systematically, collecting the terms when one expands out the brackets in eqn (5) into the terms involving no powers of a_i (and hence cubic in x_i); the terms linear in a_i (and hence quadratic in x_i); the terms quadratic in a_i (and hence linear in x_i); and the terms cubic in a_i (and hence with no powers of x_i).

It is the terms in the second of these four classes that are just a little tricky. One needs to show that these terms, which are quadratic in x_i, can actually be expressed purely in terms of the **tracefree** quadratics $(3x_i\,x_j - x_kx_k\,\delta_{ij})$, since only then will it be possible to express these terms, after integration, purely in terms of the quadrupole moment tensors Q_{ij}. A helpful way of tackling this is to define the tracefree tensor

$$S_{ij} \equiv 3x_i\,x_j - x_kx_k\,\delta_{ij} \tag{6}$$

and then use this to write

$$x_i\,x_j = \tfrac{1}{3}(S_{ij} + x_kx_k\,\delta_{ij})\,. \tag{7}$$

Now, taking your expression for all the terms linear in a_i (and hence quadratic in x_i), replace every occurrence of $x_i\,x_j$ by using (7). If you are careful with your calculations, you should find that "miraculously," all the left-over trace terms add up to zero, and thus the integral of this whole set of terms linear in a_i can be written purely in terms of Q_{ij} and a_i.

Getting this calculation to work is a good test of one's proficiency at using index notation for tensor analysis. It is totally straightforward if done correctly. But any mistakes such as clashes of dummy indices will lead to a mess and will stop it from working properly.

A general remark: It should be obvious in the explanation above but just to make it absolutely clear: when I speak, for example, of replacing "every occurrence of $x_i\,x_j$ by using (7)," I don't just mean doing this in a term where the indices happen to be called i and j! Do it also if the indices are called i and k or if they are called i and ℓ, or whatever. (And, be careful because you will have to rename the dummy k index in (7) as something else if you are wanting to use (7) to replace, for example, $x_i\,x_k$!)

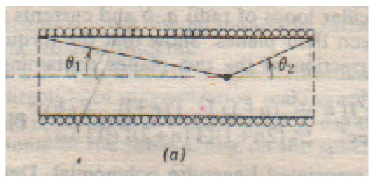
603: Electromagnetic Theory I

Problem Sheet 10

(**1a**) Consider a solenoid of radius a wound with N turns per unit length, carrying a current I. Suppose the z-axis runs along the axis of the cylinder, and that it is of length L, with $-\frac{1}{2}L \le z \le \frac{1}{2}L$. Show that when N is large, the magnetic field on the axis is in the z direction, with

$$B_z = \frac{2\pi NI}{c} \left(\cos\theta_1 + \cos\theta_2\right),$$

where θ_1 and θ_2 are indicated in the diagram below:



(**1b**) For a long solenoid where $L >> a$, show that near the axis and near the midpoint of the solenoid (i.e. $\rho << a$ and $|z| << L/2$, in cylindrical polar coordinates), the magnetic field is mainly parallel to the z axis, but has a small radial component

$$B_\rho \approx \frac{96\pi NI}{c} \frac{a^2 z\rho}{L^4}.$$

(Hint: Use $\vec\nabla \cdot \vec B = 0$.)

(**1c**) Show that near the end of a long solenoid, the magnetic field near the axis has

$$B_z \approx \frac{2\pi NI}{c}, \qquad B_\rho \approx \pm\frac{\pi NI}{c}\frac{\rho}{a}.$$

(**2a**) A circular current loop of radius a carrying a current I lies in the $x-y$ plane, centred on the origin. Show that the only non-vanishing component of the vector potential, in cylindrical polar coordinates, is

$$A_\varphi(\rho, z) = \frac{4Ia}{c} \int_0^\infty dk\, \cos kz\, I_1(k\rho_<)\, K_1(k\rho_>),$$

where $\rho_<$ and $\rho_>$ denote respectively the lesser and greater of ρ and a. (Use the expression in eqn (8.36) of the online lectures for A_φ, and the expansion in eqn (5.82) for $|\vec r - \vec r\,'|^{-1}$.)

TURN OVER FOR REMAINING QUESTIONS!!! ....

(**2b**) Show that another way of writing the vector potential is

$$A_\varphi(\rho, z) = \frac{2\pi I a}{c} \int_0^\infty dk\, e^{-k|z|}\, J_1(ka) J_1(k\rho)\,.$$

You may use the expansion

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\varphi-\varphi')}\, J_m(k\rho)\, J_m(k\rho')\, e^{-k|z-z'|}\,,$$

which can be proved (no need to do it here!) using the same methods used in the previous examples discussed in the class and in a previous homework.

(**3a**) Use suffix notation and the ϵ_{ijk} tensor to prove the Cartesian vector identities

$$\begin{array}{ll} (1) & \vec{\nabla} \times \vec{\nabla}\phi \equiv 0 \qquad \text{for any function } \phi(\vec{r})\,, \\ (2) & \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) \equiv 0 \qquad \text{for any vector } \vec{A}(\vec{r}) \\ (3) & \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \qquad \text{for any vector } \vec{A}(\vec{r}) \\ (4) & \partial_i V_j - \partial_j V_i = \epsilon_{ijk}\, W_k\,, \quad \text{where } \vec{W} = \vec{\nabla} \times \vec{V}\,. \end{array}$$

Note: The ϵ_{ijk} tensor, and its use for constructing the vector cross product in index notation, is described in section 8.3 of the online lecture notes.

Due on Tuesday 28th April (submit single PDF to grader by e-mail)

603: Electromagnetic Theory I

Commentary on Midterm 2 test

(1) Calculation of energy in the volume between concentric spherical shells

Qu. (1a) was just a standard calculation of the electrostatic energy in a volume V bounded by surface S. Using the fact that the charge density vanishes in V, an integration by parts establishes that

$$U = -\frac{1}{8\pi} \int_S \phi \, \vec{E} \cdot d\vec{S} \,. \tag{1}$$

In Qu. (1b), one considers the volume V between concentric spherical shells at radii $r = a$ and $r = b$, where $b > a$. The boundary conditions are

$$\phi(a, \theta, \varphi) = v_0 \sin\theta \cos\varphi \,, \qquad \phi(b, \theta, \varphi) = 0 \,, \tag{2}$$

and it is given that the general solution of Laplace's equation can be written as

$$\phi(r, \theta, \varphi) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} (A_{\ell m} \, r^{\ell} + B_{\ell m} \, r^{-\ell-1}) \, Y_{\ell m}(\theta, \varphi) \,. \tag{3}$$

It is also given that

$$Y_{1,1}(\theta, \varphi) = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\mathrm{i}\varphi} \,, \quad Y_{1,-1}(\theta, \varphi) = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-\mathrm{i}\varphi} \,, \quad Y_{1,0}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta \,.$$

In a problem like this, it is a good idea to look first at how the boundary conditions can be written in terms of the spherical harmonics, because this will dictate which harmonics arise in the solution. It is always useful to bear in mind that if the potential on all the boundaries was zero, the answer for the potential $\phi(r, \theta, \varphi)$ inside the volume V would simply be zero. The only thing that makes ϕ non-zero in V is the non-vanishing of ϕ on the boundary. And since the different modes in the harmonic expansions operate independently of one another, the only spherical harmonics that arise in the expansion for ϕ in V will be those that arise in the boundary conditions.

In a general such problem, the functional form of the potentials $\phi(a, \theta, \varphi)$ and $\phi(b, \theta, \varphi)$ on the boundary surfaces would require expansions involving sums over an infinity of terms, involving all the spherical harmonics. One then needs the whole machinery of orthogonality relations and so on, in order to obtain expressions for the coefficients in the expansion as integrals over the boundaries. But in many examples one encounters in tests and homeworks, the setter has been kind and picked boundary potentials that involve just a finite number of terms, and one can see by inspection how to write them in terms of just a few of the $Y_{\ell m}$.

In the present case, we see that the boundary potential $\phi(a,\theta,\varphi)$ can be written in terms of just $Y_{1,1}$ and $Y_{1,-1}$.

In fact we can now really make life very simple. Because the two relevant modes both have $\ell = 1$ we know from (3) that the r dependence will be either r or $1/r^2$. So without further ado we can say that the relevant solution of Laplace's equation that we are seeking must simply take the form

$$\phi(r,\theta,\varphi) = \Big(\alpha\, r + \frac{\beta}{r^2}\Big)\, \sin\theta\, \cos\varphi\,, \tag{4}$$

where α and β are constants that will be fixed by matching to the boundary conditions in eqn (2). Nobody obliges us to use the standard $Y_{\ell m}(\theta,\varphi)$ basis of modes for the spherical harmonics. They are convenient when discussing things in generalities, but in a specific example like the present one, they are just a pain in the neck, with all their $\sqrt{1/\pi}$, etc., factors, and the use of the complex exponentials. For $\ell = 1$ modes, a basis for the three independent spherical harmonics can equally well be taken to be the linear combinations of $Y_{1,1}$, $Y_{1,-1}$ an $Y_{1,0}$ that are $\sin\theta\, \cos\varphi$, $\sin\theta\, \sin\varphi$ and $\cos\theta$. (Or, in other words, x/r, y/r and z/r, as was pointed out earlier in the lectures.)

Anyway, going back to our present problem we just have to solve for α and β in (4) by requiring the two equations in (2) to hold. With minimal algebra we arrive at

$$\phi(r,\theta,\varphi) = \alpha\, \Big(r - \frac{b^3}{r^2}\Big)\, \sin\theta\, \cos\varphi\,, \qquad \alpha = \frac{v_0\, a^2}{a^3 - b^3}\,. \tag{5}$$

Note that this is the form that was requested in the question; it asked for the answer to be expressed in terms of trigonometric functions of θ and φ, rather than in terms of the $Y_{\ell m}$. Of course one could do this by first getting the answer in terms of $Y_{1,1}$ and $Y_{1,-1}$, and then finally converting back to get the $\sin\theta\, \cos\varphi$. But, as pointed out above, the form of the boundary condtions in this problem was so simple that we could side-step the necessity of ever going into the gory details of the expressions using $Y_{\ell m}$'s. Instead, we could just use the argument that led directly to writing the solution in the form of eqn (4).

Having obtained the solution in the form of (5) it is now easy enough in Qu. (1c) to plug into the expression (1) to calculate the electrostatic energy in V. Since $\phi(b,\theta,\varphi) = 0$ this comes entirely from the integral over the $r = a$ surface. The only slight subtlety to watch out for is that since the boundary area element $d\vec{S}$ points *outwards* from the volume V, at the $r = a$ surface it points in the direction of *decreasing* r. This ensures that the energy comes out to be positive, and one staightforwardly finds

$$U = \frac{v_0^2\, a\, (a^3 + 2b^3)}{6(b^3 - a^3)}\,. \tag{6}$$

(2) Potential for a dielectric sphere with an external point charge

Not too much to comment on in Qu. (2a). It is just a standard little calculation to get the expressions for the potential $q\,|\vec{r}-d\,\hat{z}|^{-1}$ due to a point charge q on the z axis at $z=d$ (where $\hat{z}=(0,0,1)$ is the unit vector along z), as expansions in Legendre polynomials:

$$\phi_<(r,\theta) = q\sum_{\ell\ge 0}\frac{r^\ell}{d^{\ell+1}}\,P_\ell(\cos\theta)\,,\qquad \text{valid in the region } r<d\,,$$

$$\phi_>(r,\theta) = q\sum_{\ell\ge 0}\frac{d^\ell}{r^{\ell+1}}\,P_\ell(\cos\theta)\,,\qquad \text{valid in the region } r>d\,.$$

Of course we know the closed-form result $\phi(r,\theta)=q\,(r^2-2rd\cos\theta+d^2)^{-1/2}$ already, but the expansions are needed for the later parts of the problem.

In Qu. (2b), a dielectric sphere of radius $r=a$ (less than d) is introduced, and one is directed to solve the problem by superposing on the previous pure-vacuum solution the general expansions

$$\begin{aligned}\phi_-(,r,\theta) &= \sum_{\ell\ge 0} A_\ell\, r^\ell\, P_\ell(\cos\theta)\,, &&\text{valid for } r\le a\,,\\ \phi_+(,r,\theta) &= \sum_{\ell\ge 0} B_\ell\, r^{-\ell-1}\, P_\ell(\cos\theta)\,, &&\text{valid for } r\ge a\,.\end{aligned}$$

Thus there will now be three regions, with

$$\begin{aligned}\phi_1 &= \phi_- + \phi_<\,, &&\text{for} && 0\le r\le a\,;\\ \phi_2 &= \phi_+ + \phi_<\,, &&\text{for} && a\le r< d\,;\\ \phi_3 &= \phi_+ + \phi_>\,, &&\text{for} && r>d\,.\end{aligned}\tag{7}$$

The coefficients A_ℓ and B_ℓ can be solved for by imposing the boundary conditions at the $r=a$ junction. Namely, the normal component of $\vec{D}$ is continuous at $r=a$, and the potential is continuous at $r=a$.[1] Thus, the two boundary conditions are

$$\epsilon\frac{\partial\phi_1}{\partial r}\Big|_{r=a} = \frac{\partial\phi_2}{\partial r}\Big|_{r=a}\,,\qquad \phi_1|_{r=a}=\phi_2|_{r=a}\,.\tag{8}$$

[1] As discussed in the lectures, a common way of stating the second type of boundary condition is that the tangential components of $\vec{E}$ are continuous at the junction. But this is really just a (slightly weaker) way of saying that ϕ must be continuous at the junction. The extra little bit of information contained in saying that ϕ itself must be continuous at the junction is the fact that the values of ϕ on the two sides of the junction must literally be equal, and they cannot differ even by a pure additive constant. This continuity of the potential across the junction must in fact clearly always hold, since if it did not, there would be a step-function in the potential as a function of the coordinate normal to the boundary (r in the present example under discussion). If such a step function existed then it would mean a delta function in the normal gradient of the potential; i.e. a delta function in the normal component of the electric field at the junction. This would be unphysical, and in conflict with conditions assumed when solving the problem.

Note that because of the forms of ϕ_1 and ϕ_2 in eqns (7), the $\phi_<$ term cancels between the two sides in the second condition, and one just has $\phi_-|_{r=a} = \phi_+|_{r=a}$. But the $\phi_<$ term does not cancel between the two sides in the first condition, because there is an ϵ factor on the left-hand side but not on the right-hand side. One straightforwardly solves the two conditions, for each ℓ, to give

$$A_\ell = \frac{(1-\epsilon)\, q\, \ell}{(\epsilon\, \ell + \ell + 1)\, d^{\ell+1}}\,, \qquad B_\ell = \frac{(1-\epsilon)\, q\, \ell\, a^{2\ell+1}}{(\epsilon\, \ell + \ell + 1)\, d^{\ell+1}}\,. \tag{9}$$

With these solved for, we have the full solutions for the potentials ϕ_1, ϕ_2 and ϕ_3 in (7) in the three regions.

One remark is worth making here. Notice that we *cannot* simply solve this problem by trying a simple "method of images" kind of ansatz for the potential. The answer, as obtained above, cannot be written as simply the sum of the potential due to the original point charge q at $z = d$ plus the potential of another "image" point charge at some other location. The key thing that demonstrates this is that the coefficients in the expansions over the Legendre polynomials $P_\ell(\cos\theta)$ have ℓ-dependent coefficients multiplying the $d^{-\ell-1}$ and/or $a^{2\ell+1}$ factors (see 9)). This would not happen if one had simple expansions for isolated point charges at specific locations.

I might add at this point that there are some alleged "solutions" to this dielectric sphere problem circulating on the internet that claim that it can be solved as a method of images problem. I don't want to interrupt the thread of this commentary here by diverting into discussing a wrong claim, but I'll add a short supplementary comment at the end of this document.

The story changes if we take the limit where ϵ goes to infinity, as in Qu. (2c). In the limit one has

$$A_\ell \longrightarrow -\frac{q}{d^{\ell+1}}\,, \qquad B_\ell \longrightarrow -\frac{q\, a^{2\ell+1}}{d^{\ell+1}}\,, \tag{10}$$

and so we no longer have ℓ-dependence in the factors multiplying the $d^{-\ell-1}$ or $a^{2\ell+1}$. If we define

$$q' = -\frac{a\, q}{d}\,, \qquad b = \frac{a^2}{d}\,, \tag{11}$$

then the expression for B_ℓ in (10) becomes

$$B_\ell = q'\, b^\ell\,, \tag{12}$$

and the potential ϕ_+ becomes

$$\phi_+(r,\theta) = q' \sum_{\ell\ge 0} \frac{b^\ell}{r^{\ell+1}}\, P_\ell(\cos\theta)\,, \tag{13}$$

which can be recognised as

$$\phi_+(r,\theta) = \frac{q'}{\sqrt{r^2 - 2br\cos\theta + b^2}}\,, \qquad (14)$$

which is the potential of an "image charge" $q' = -aq/d$ located at $z = b = a^2/d$ on the z axis. Thus, in this $\epsilon \to \infty$ limit, where the sphere behaves like a solid conducting grounded ball, the potential outside is indeed just given by the usual result from the method of images.

Comment on an incorrect "solution" to the dielectric sphere problem:

One can find on the internet a claimed derivation of the potential due to a point charge outside a dielectric sphere that purports to treat it as a method of images problem, in which the potentials outside and inside the sphere are written as

$$\begin{aligned}\Phi_{\rm out} &= \frac{q}{|\vec r - d\,\hat z|} + \frac{q'}{|\vec r - (a^2/d)\,\hat z|}\,, \qquad r \ge a\,,\\ \Phi_{\rm in} &= \frac{q''}{|\vec r - d\,\hat z|}\,, \qquad r \le a\,,\end{aligned} \qquad (15)$$

where $\hat z \equiv (0,0,1)$ is the unit vector along z. (See, for example, link .) The claim that the potential can be written in this way is incorrect, however. This can be easily seen by looking further at the claimed "proof." Reading on in the "derivation," one finds that the two constants q' and q'' in eqn (15) have mysteriously become dependent on the mode number ℓ after the author has expanded in the Legendre polynomials $P_\ell(\cos\theta)$. This is completely meaningless. If one looks at eqn (15), there is no concept of ℓ in those formulae (since no expansion in Legendre polynomials has yet been made), and so no meaning can be attached to the claim that q' and q'' depend on ℓ. The author has actually blundered through to correct final expressions for the potential by starting with an incorrect assumption at the beginning, and then sneaking in some "mid-course corrections" in the middle in order to change the problem into something that does work. But in the process, the logic is completely distorted, or indeed, absent. The correct answer for the potential in this problem is not of the form (15), and thus a claimed derivation that takes (15) as its starting point is not valid. Steer clear of this "approach"!

In fact, the it is instructive to write the actual solutions for the interior and exterior potentials in a style somewhat analogous to the incorrect expressions in (15). We have

$$\begin{aligned}\phi_{\rm out} &= \frac{q}{|\vec r - d\,\hat z|} + \phi_+ = \frac{q}{|\vec r - d\,\hat z|} + \sum_{\ell\ge0} \frac{B_\ell}{r^{\ell+1}}\,P_\ell(\cos\theta)\,, \qquad r \ge a\\ \phi_{\rm in} &= \frac{q}{|\vec r - d\,\hat z|} + \phi_- = \frac{q}{|\vec r - d\,\hat z|} + \sum_{\ell\ge0} A_\ell\, r^\ell\,P_\ell(\cos\theta)\,, \qquad r \le a\end{aligned} \qquad (16)$$

where the A_ℓ and B_ℓ are given in eqn (9). (The expression for $\phi_{\rm out}$ is just the same as ϕ_2 when $a \le r < d$ and the same as ϕ_3 when $r > d$. The expression for $\phi_{\rm in}$ is the same as ϕ_1 when $r \le a$.)

Crucially, neither of the expressions ϕ_+ or ϕ_-, i.e.

$$\begin{aligned} \phi_+(r,\theta) &= \sum_{\ell\ge 0} \frac{(1-\epsilon)\, q\, \ell\, a^{2\ell+1}}{(\epsilon\,\ell+\ell+1)\, d^{\ell+1}\, r^{\ell+1}}\, P_\ell(\cos\theta)\,, \\ \phi_-(r,\theta) &= \sum_{\ell\ge 0} \frac{(1-\epsilon)\, q\, \ell\, r^{\ell}}{(\epsilon\,\ell+\ell+1)\, d^{\ell+1}}\, P_\ell(\cos\theta)\,, \end{aligned} \tag{17}$$

is the potential due to a point charge located at any single point.

As we saw earlier, in the limit $\epsilon \to \infty$ the expression ϕ_+ *does* become the potential of an image charge $-qa/d$ located at $z = a^2/d$. Only in this limit does the usual method of images work.

603: Electromagnetic Theory I

Commentary on Final exam

Question (1)

Qu. (1a) involves solving for the potential everywhere outside the grounded sphere of radius $r = a$, coated with a shell of dielectric material of dielectric constant ϵ out to a radius $r = b$. Everywhere outside $r = b$ is a vacuum, and the question states that the electric field becomes asymptotically constant at large distances, with $\vec{E} \to (0, 0, E_0)$ (these are the Cartesian components). Of course, we choose to use the standard spherical polar coordinates, and we know there is azimuthal symmetry around the z-axis. Thus, we know that at large r the potential $\phi(r, \theta)$ approaches $\phi(r, \theta) \to -E_0\, r\, \cos\theta$.

The question contains the reminder that the general azimuthally-symmetric solution of Laplace's equation can be expanded in the form

$$\phi(r,\theta) = \sum_{\ell\ge 0} (A_\ell\, r^\ell + B_\ell\, r^{-\ell-1})\, P_\ell(\cos\theta)\,. \tag{1}$$

As stated in the question, one needs to write ϕ using two such expansions, to cover the regions $a \le r \le b$ and $r \ge b$:

$$\phi_1(r,\theta)\,, \qquad \text{for} \quad a \le r \le b\,,$$
$$\phi_2(r,\theta)\,, \qquad \text{for} \quad r \ge b\,.$$

The boundary conditions $\phi(a, \theta) = 0$ and $\phi(r, \theta) \to -E_0\, r\, \cos\theta$ at large r, together with the junction conditions at the dielectric interface at $r = b$, will then allow one to solve completely for the coefficients in the ϕ_1 and ϕ_2 expansions.

The question also included the reminder that one can simplify life by pausing to look at which modes in the expansions for ϕ_1 and ϕ_2 of the form in eqn (1) will actually arise in the solution. This is a rather important point, because, especially in the situation of a time-limited test or exam, anything one can do to reduce the amount of calculation is going to be useful. There are two points that should be emphasised here:

A: We are solving linear equations here (the Maxwell equations are linear in the electric and magnetic fields), and so that means that when the scalar potential is expanded in the form (1), the modes associated with the different ℓ values *do not interact with each other.* **Each value of ℓ in the expansion can be discussed independently of each other value of ℓ.**

Now in the present example, the entire solution is being driven by that asymptotic form of the scalar potential; $\phi(r, \theta) \to -E_0\, r\, \cos\theta$ as r goes to infinity. In particular, if E_0

happened to be zero, the entire solution for the potential outside the sphere would simply be $\phi(r,\theta)=0$. The **only** thing that causes the potential to be non-zero in the solution is that non-vanishing asymptotic boundary condition. Since $\cos\theta$ is just the $\ell=1$ Legendre polynomial, $P_1(\cos\theta)=\cos\theta$, this means therefore that in the entire problem, we only need the $\ell=1$ modes. That is to say, not only in the expansion for $\phi_2(r,\cos\theta)$ in the $r\ge b$ region, but also in the expansion for $\phi_1(r,\cos\theta)$ in the $a\le r\le b$ region, we only need the $\ell=1$ modes. So we can write

$$\phi_1(r,\theta)=(\alpha\, r+\beta\, r^{-2})\,\cos\theta\,,\qquad \phi_2(r,\theta)=(-E_0\, r+\gamma\, r^{-2})\,\cos\theta\,, \tag{2}$$

for constant α, β and γ that will be fixed by the remaining boundary and junction conditions. (We have already fixed the value of the constant in the first term in ϕ_2 by using the asymptotic boundary condition $\phi(r,\theta)\to -E_0\, r\,\cos\theta$ at large r.)

There is nothing actually wrong, of course, with writing down completely general expansions of the form (1) for each of ϕ_1 and ϕ_2, with the whole infinity of modes included. But there is absolutely no need to include them, for the reasons observed previously; only $\ell=1$ modes can actually be involved in this solution. All the other modes will turn out to vanish once the boundary conditions are imposed. But including them in the calculations increases the risk of getting bogged down in a morass of equations, and one may also end up making some errors in the general-ℓ calculations that persist in the relevant $\ell=1$ subset too. Spending a few moments at the beginning of the calculation in pruning the expansions down to the tiny subset of terms that will actually arise is usually very well worthwhile.

One only has a finite amount of time and energy available for calculating, so one should avoid wearing oneself down on long and involved calculations with terms that can in fact be easily seen from the outset to be irrelevant to the problem! Getting bogged down in a lot of algebra for the general-ℓ modes, leading to running out of time and/or making mistakes, was overwhelmingly the leading reason for the difficulties some people ran into when doing this exam.

We have now pared the calculation down to imposing the remaining boundary and junction conditions on the expressions (2), namely

$$\phi_1(a,\theta)=0\,,\qquad \epsilon\,\frac{\partial\phi_1(r,\theta)}{\partial r}\Big|_{r=b}=\frac{\partial\phi_2(r,\theta)}{\partial r}\Big|_{r=b}\,,\qquad \phi_1(b,\theta)=\phi_2(b,\theta)\,. \tag{3}$$

It is straightforward to solve the resulting three equations for the three unknowns α, β and γ, giving

$$\alpha=-\frac{3b^3\,E_0}{2(b^3-a^3)+\epsilon\,(2a^3+b^3)}\,,\qquad \beta=-\alpha\, a^3\,,\qquad \gamma=\alpha\,(b^3-a^3)+b^3\,E_0\,. \tag{4}$$

In case any doubts remain about the validity of the arguments above for paring the problem just down to its essence, namely the $\ell = 1$ terms only, remember the second very important point:

B: The uniqueness theorem assures us that there exists one, and only one, solution to a well-posed boundary-value problem in electrostatics. So provided we are able to solve Laplace's equation (and the expressions in (2) certainly do that), and provided that we are able to satisfy the boundary conditions (and we certainly can, with α, β and γ given by (4)), then we are *guaranteed* that we have correctly solved the problem.

Qu. (1b) is concerned with calculating the electrostatic energy in the shell of dielectric medium. This could be done by directly calculating the the quantity $\vec{E} \cdot \vec{D}$ appearing in the energy integral $u = 1/(8\pi) \int_V (\vec{E} \cdot \vec{D})\, d^3\vec{r}$, and then integrating it over the 3-volume of the dielectric shell. But the calculation is considerably simplified by first turning it into a surface integral, by means of an integration by parts, so that one has

$$U_{\rm dielectric} = \frac{\epsilon}{8\pi} \int_{S_b} \phi_1 \, \frac{\partial \phi_1}{\partial r}\, dS\Big|_{r=b} \,. \tag{5}$$

(The integral is only over the outer surface at $r = b$, since ϕ_1 vanishes on the inner surface at $r = a$.)

Plugging the expression for ϕ_1 (best left in the form $\phi_1(r,\theta) = \alpha\,(r - a^3\, r^{-2})\,\cos\theta$ until the very last line of the calculation, when α can be substituted in using (4)) straightforwardly then gives

$$U_{\rm dielectric} = \frac{3\epsilon\, E_0^2\, b^3\,(b^3 - a^3)(2a^3 + b^3)}{2[2(b^3 - a^2) + \epsilon\,(2a^3 + b^3)]^2} \,. \tag{6}$$

(It is not a good idea to write the explicit expressions for α and β in every step of the calculation. This not only costs time, but also increases the risk of making errors.)

Qu. (1c) asks you to show that if a, b and E_0 are held fixed, while allowing ϵ to vary, there is in fact a maximum value of $U_{\rm dielectric}$ that is achieved. The question asks you to calculate the value $\epsilon_{\rm max}$ that achieves this maximisation, and also to calculate the corresponding value $U_{\rm max}$ of the energy at the maximum.

The first step is to take the result in (6) for the energy $U_{\rm dielectric}$, differentiate with respect to ϵ, and solve for $\partial U_{\rm dielectric}/\partial\epsilon = 0$. The question suggested to take the expression in (6), and write it as

$$U_{\rm dielectric} = \frac{A\,\epsilon}{(B + C\,\epsilon)^2} \,, \tag{7}$$

so we see that

$$A = \frac{3\epsilon\, E_0^2\, b^3\,(b^3 - a^3)(2a^3 + b^3)}{2} \,, \qquad B = 2(b^3 - a^3)\,, \qquad C = 2a^3 + b^3\,. \tag{8}$$

The point of the suggestion was simply to make the intermediate steps of algebra easier, thus reducing the risk of making mistakes that could arise when carrying forward complicated numerators and denominators from one line to the next in a calculation. The final answers for $\epsilon_{\rm max}$ and $U_{\rm max}$ should, of course, be presented in terms of the parameters a, b, E_0 and ϵ of the original specification of the problem, but using A, B and C in the intermediate stages is quite helpful for keeping the algebra simple.

Thus from eqn (7) we easily see that solving $\partial U_{\rm dielectric}/\partial\epsilon = 0$ for ϵ gives

$$\epsilon = \epsilon_0 \equiv \frac{B}{C}\,. \tag{9}$$

We call it ϵ_0 for now, not predjucing whether it corresponds to a maximum, a minimum, or a point of inflection for $U_{\rm dielectric}$. We can address this question right away: Noting from (8) that A, B and C are positive quantities we see that $U_{\rm dielectric}$ in (7) is zero when $\epsilon = 0$ and it increases as ϵ goes positive. Since there is only the one stationary point (at finite ϵ), corresponding to the single solution (9) for $\partial U_{\rm dielectric}/\partial\epsilon = 0$, and it occurs for a positive ϵ, it must therefore be that it corresponds to a maximum. (Of course one could also show it is a maximum by calculating $\partial^2 U_{\rm dielectric}/\partial\epsilon^2$ at $\epsilon = \epsilon_0$, finding $(\partial^2 U/\partial^2\epsilon)|_{\epsilon=\epsilon_0} = -AC/(8B^3)$, which is negative, but that seems like slightly more work than the argument above.)

Now, plugging $\epsilon = \epsilon_0$ into (7) gives the corresponding value of U which is

$$U\Big|_{\epsilon=\epsilon_0} = \frac{A}{4BC}\,. \tag{10}$$

Finally, we substitute for A, B and C, as defined in (8), finding that we have

$$\epsilon_{\rm max} = \frac{2(b^3-a^3)}{2a^3+b^3}\,, \qquad U_{\rm max} = \frac{3}{16}\, b^3\, E_0^2\,. \tag{11}$$

Notice that some nice cancellations of numerator and denominator factors have occurred in the final expression for $U_{\rm max}$.

Question (2)

Question (2a) asked you to find current density $\vec J$, for a rotating hollow cylinder of radius a and height $2h$, located coaxially with the z axis and extending from $z=-h$ to $z=h$. Some discussion showing how the given Cartesian coordinate expression

$$\vec J(\vec r\,) = \sigma\, a\, \omega\, \delta(\rho-a)\, (-\sin\varphi, \cos\varphi, 0)\,, \qquad \text{for} \quad -h \le z \le h \tag{12}$$

comes about, by considering the rate of flow of charge in the sheet on the surface of the rotating cylinder, is what was really wanted here.

In Question (2b), the expression for $\vec{J}$ from qu. (2a) is used in order to calculate the vector potential, using the Cartesian coordinate result

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'\,. \tag{13}$$

It is given in the question that the Green function $|\vec{r}-\vec{r}\,'|^{-1}$ can be written in cylindrical polar coordinates as

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{\mathrm{i}m(\varphi-\varphi')}\, J_m(k\rho)\, J_m(k\rho')\, e^{-k|z-z'\,|}\,. \tag{14}$$

The question asks first for the Cartesian components of $\vec{A}(\vec{r})$. Thus one plugs (12) into (13) (remembering, of course, that one must put primes on the coordinates ρ and φ in (12)), making use of (14). Let's look and some of the key elements involved in the calculation:

First, consider the φ' integration. This will involve evaluating

$$\int_0^{2\pi} d\varphi'\, e^{-\mathrm{i}m\varphi'}\, \sin\varphi' \qquad \text{and} \qquad \int_0^{2\pi} d\varphi'\, e^{-\mathrm{i}m\varphi'}\, \cos\varphi'\,. \tag{15}$$

Using $\sin\varphi' = \frac{1}{2\mathrm{i}}\,(e^{\mathrm{i}\varphi'} - e^{-\mathrm{i}\varphi'})$ and $\cos\varphi' = \frac{1}{2}\,(e^{\mathrm{i}\varphi'} + e^{-\mathrm{i}\varphi'})$, together with

$$\int_0^{2\pi} d\varphi'\, e^{\mathrm{i}(n-m)\varphi'} = 2\pi\, \delta_{m,n}\,, \tag{16}$$

we obtain

$$\int_0^{2\pi} d\varphi'\, e^{-\mathrm{i}\varphi'}\, \sin\varphi' = -\mathrm{i}\pi\,(\delta_{m,1} - \delta_{m,-1})\,, \qquad \text{and} \qquad \int_0^{2\pi} d\varphi'\, e^{-\mathrm{i}\varphi'}\, \cos\varphi' = \pi\,(\delta_{m,1} + \delta_{m,-1})\,. \tag{17}$$

Noting that, as given in the question, $J_{-m}(x) = (-1)^m\, J_m(x)$, we see that $J_{-1}(k\rho)J_{-1}(k\rho') = J_1(k\rho)J_1(k\rho')$, and so we shall have

$$\begin{aligned}\sum_{m=-\infty}^{\infty}\int_0^{2\pi} d\varphi'\, e^{\mathrm{i}m(\varphi-\varphi')}\, J_m(k\rho)J_m(k\rho')\, \sin\varphi' &= -\mathrm{i}\pi(e^{\mathrm{i}\varphi} - e^{-\mathrm{i}\varphi})\, J_1(k\rho)J_1(k\rho') \\ &= 2\pi\, J_1(k\rho)J_1(k\rho')\, \sin\varphi \end{aligned} \tag{18}$$

and

$$\begin{aligned}\sum_{m=-\infty}^{\infty}\int_0^{2\pi} d\varphi'\, e^{\mathrm{i}m(\varphi-\varphi')}\, J_m(k\rho)J_m(k\rho')\, \cos\varphi' &= \pi(e^{\mathrm{i}\varphi} + e^{-\mathrm{i}\varphi})\, J_1(k\rho)J_1(k\rho') \\ &= 2\pi\, J_1(k\rho)J_1(k\rho')\, \cos\varphi\,. \end{aligned} \tag{19}$$

Now consider the z' integration. This will require evaluating

$$\int_{-h}^{h} dz'\, e^{-k|z-z'|}\,. \tag{20}$$

The only subtlety here comes from handling the $|z - z'|$. Luckily, the person setting the exam was kind enough to ask for the expression for $\vec{A}(\vec{r})$ only in the region where $z > h$. This is nice because it means that z is greater than any of the values that z' will take in the integration from $z' = -h$ up to $z' = +h$, and so we shall simply have

$$\begin{aligned} \int_{-h}^{h} dz'\, e^{-k|z-z'|} &= \int_{-h}^{h} dz'\, e^{-k(z-z')} = e^{-kz} \int_{-h}^{h} dz'\, e^{kz'} = \frac{1}{k}\, e^{-kz}\, (e^{kh} - e^{-kh}) \\ &= \frac{2\sinh kh}{k}\, e^{-kz}\,. \end{aligned} \tag{21}$$

(If nothing had been stated about the value of z, there would have been tedious complications for the cases where z took values between $-h$ and $+h$, in which the z' integration would need to be split into the portion where $|z - z'| = z - z'$ and the portion where $|z - z'| = z' - z$.)

Putting everything together, we see that the final result for the Cartesian components of $\vec{A}$ is

$$\vec{A}(\vec{r}) = \frac{4\pi a^2\, \sigma\, \omega}{c}\, (-\sin\varphi, \cos\varphi, 0) \int_0^\infty dk \frac{\sinh kh}{k}\, e^{-kz}\, J_1(k\rho)\, J_1(ka)\,. \tag{22}$$

Hence finally, noting that $A_x = -A_\varphi \sin\varphi$ and $A_y = A_\varphi \cos\varphi$, we see that in terms of its components in cylindrical polar coordinates we have

$$A_\varphi = \frac{4\pi a^2\, \sigma\, \omega}{c} \int_0^\infty dk \frac{\sinh kh}{k}\, e^{-kz}\, J_1(k\rho)\, J_1(ka)\,, \tag{23}$$

together with $A_\rho = 0$ and $A_z = 0$.

It is perhaps worth remarking that this result, which is giving the exact vector potential for a uniform sheet of current flowing around a finite-length cylinder, gives also, with an appropriate re-interpretation of the constants, the exact expression for the potential due to an idealised finite-length solenoid. (That is, for a finite-length solenoid where the number of turns per unit length is sufficiently great that one effectively has a uniform sheet of current flowing around the solenoid.)